

Improving the bound for maximum degree on Murty-Simon Conjecture

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Abstract

A graph is said to be diameter- k -critical if its diameter is k and removal of any of its edges increases its diameter. A beautiful conjecture by Murty and Simon, says that every diameter-2-critical graph of order n has at most $\lfloor n^2/4 \rfloor$ edges and equality holds only for $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$. Haynes et al. proved that the conjecture is true for $\Delta \geq 0.7n$. They also proved that for $n > 2000$, if $\Delta \geq 0.6789n$ then the conjecture is true. We will improve this bound by showing that the conjecture is true for every n if $\Delta \geq 0.6755n$.

1 Introduction

Throughout this paper we assume that G is a simple graph. Our notation is the same as [3], let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size m . For a vertex $v \in G$ we denote the set of its neighbors in G by $N_G(v)$. Also we denote $N_G(v) \cup v$ by $N_G[v]$. The maximum and minimum degrees of G will be denoted by Δ and δ , respectively. The distance $d_G(u, v)$ between two vertices u and v of G , is the length of the shortest path between

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them. The *diameter* of G , ($\text{diam}(G)$), is the maximum distance among all pairs of vertices in G .

We say graph G is *diameter- k -critical* if its diameter is k and removal of any of its edges increases its diameter. Based on a conjecture proposed by Murty and Simon [5], there is an upper bound on the number of edges in a diameter-2-critical graph.

Conjecture 1.1. *Let G be a diameter-2-critical graph. Then $m \leq \lfloor n^2/4 \rfloor$ and equality holds only if $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$.*

Several authors have conducted some studies on the conjecture proving acceptable results nearly close to the original one, however, no complete proof has been provided yet. Plesnk [6] showed that $m < \frac{3n(n-1)}{8}$. Moreover, Caccetta and Haggkvist [5] proved $m < 0.27n^2$. Fan [7] also proved the fact that for $n \leq 24$ and for $n = 26$ we have $m \leq \lfloor \frac{n^2}{4} \rfloor$. For $n = 25$, he achieved $m < \frac{n^2}{4} + \frac{(n^2-16.2n+56)}{320} < 0.2532n^2$. Another proof was presented by Xu [8] in 1984, which was found out to have a small error. Afterwards, Furedi [9] provided a considerable result showing that the original conjecture is true for large n , that is, for $n > n_0$ where n_0 is a tower of 2s of height about 10^{14} . This result is highly significant though not applicable to those graphs we are currently working with.

2 Total Domination

Domination number and Total domination number are parameters of graphs which are studied, respectively, in [2, 14] and [15]. Assume $G = (V, E)$ is a simple graph. Let X and Y be subsets of V ; We say that X dominates Y , written $X \succ Y$, if and only if every element of $Y - X$ has a neighbor in X . Similarly, we say that X totally dominates Y , written $X \succ_t Y$ if and only if every element of Y has a neighbor in X . If X dominates or totally dominates V , we might write, $X \succ G$ or $X \succ_t G$ instead of $X \succ V$ and $X \succ_t V$, respectively. Domination number and total domination number of $G = (V, E)$ are the size of smallest subset of V that, dominates and totally dominates V , respectively. A graph G with total domination number of k is called k_t -critical, if every graph constructed by adding an edge between any nonadjacent vertices of G has total domination number less than k . It is obvious that adding any edge to k_t -critical graph G would result a graph which has total domination number of $k - 1$ or $k - 2$. Assume G is k_t -critical graph. If for every pair of non adjacent vertices $\{u, v\}$ of G , the total domination number of $G + uv$ is $k - 2$, then G is called k_t -supercritical. As shown in [4] there is a great connection between diameter-2-critical graphs and total domination critical graphs:

Theorem 2.1. ([4]) *A graph is diameter-2-critical if and only if its complement is 3_t -critical or 4_t -supercritical.*

By this theorem in order to prove Murty-Simon conjecture, it suffices to prove that every graph which is 3_t -critical, or 4_t -critical, has at least $\lfloor n(n -$

$2)/4]$ edges where n is order of graph. This problem is solved in some cases in [10, 11, 12] :

Theorem 2.2. ([10]) *A graph G is 4_t -supercritical if and only if G is disjoint union of two nontrivial complete graphs.*

Theorem 2.3. ([11]) *If G is a 3_t -critical graph, then $2 \leq \text{diam}(G) \leq 3$.*

Theorem 2.4. ([12]) *Every 3_t -critical graph of diameter 3 and order n has size $m \geq n(n-2)/4$.*

By this theorems a proof for following conjecture will show that Murty-Simon conjecture is true.

Conjecture 2.5. *A 3_t -critical graph of order n and of diameter 2 has size $m \geq n(n-2)/4$.*

More recently Haynes et al. proved the following:

Theorem 2.6. ([13]) *Let G be a 3_t -critical graph of order n and size m . Let $\delta = \delta(G)$. Then the following holds:*

a) If $\delta \geq 0.3n$, then $m > \lceil n(n-2)/4 \rceil$.

b) If $n \geq 2000$ and $\delta \geq 0.321n$, then $m > \lceil n(n-2)/4 \rceil$.

Also G. Fan et al. proved that:

Theorem 2.7. ([7]) *The Murty-Simon conjecture is true for every graph with less than 25 vertices.*

In next section, in order to improve this bound, we will prove that, every simple diameter-2-critical graph of order n and size m satisfies $m < \lfloor n^2/4 \rfloor$ if $\Delta \geq 0.6756n$.

3 Main Result

In this section we will prove Murty-Simon conjecture for graphs which their complement are 3_t -critical and have less restriction on their minimum degree and improve the result proposed by Haynes et al in [13]. First we recall the following lemma, which was proposed in that paper.

Lemma 3.1. *Let u and v are nonadjacent vertices in 3_t -critical graph G , clearly $\{u, v\} \not\subseteq G$. Then there exists a vertex w , such that w is adjacent to exactly one of u, v , say u , and $\{u, w\} \succ G - v$. We will call uw quasi-edge associated with uv . Further v is the unique vertex not dominated by $\{u, w\}$ in G ; In this case we call v supplement of $\{u, w\}$.*

Definition 3.1. Let $G = (V, E)$ be a 3_t -critical graph. If $S \subseteq V$ then we say that S is a *quasi-clique* if for each nonadjacent pair of vertices of S there exists a quasi-edge associated with that pair, and each quasi-edge associated with that pair at contains at least on vertex outside S . Edges *associated with* quasi-clique S are the union of the edges with both ends in S and the quasi-edges associated with some pair of nonadjacent vertices of S .

Definition 3.2. Let $G = (V, E)$ be a 3_t -critical graph. Let A and B be two disjoint subsets of V . We define $E(G; A, B)$ as set of all edges $\{a, b\}$ where $a \in A$ and $b \in B$, and $\{a, b\}$ is associated with a non adjacent pair $\{a, c\}$, where c is in A . By lemma 3.1, we know that every two members of $E(G; A, B)$ are associated with different non adjacent pairs.

Lemma 3.2. Let G be a 3_t -critical graph. Let $S \subset V(G)$, if $S^* = \cap_{s \in S} N(s)$, then the following holds:

$$|E(G[S^*])| + |E(G; S^*, V(G) - (S^* \cup S))| \geq \frac{|S^*|^2 - 2|S^*|}{c}$$

Where c is the greatest root of $x^2 - 4x - 4 = 0$, which is equal to $2 + 2\sqrt{2} \approx 4.83$.

Proof. We apply induction on size of S^* to prove the theorem. Note that for every pair of non-adjacent vertices in S^* such as $\{u, v\}$, If $\{u, w\}$ is the quasi-edge associated to it, then, since v is adjacent u , we can conclude that $w \notin S^*$. Note that when $|S^*| \leq 2$, since $\frac{|S^*|^2 - 2|S^*|}{c} \leq 0$, then the inequality is obviously true. Let v be the vertex having minimum degree in $G[S^*]$. We denote the set of neighbors of v in S^* by A . Since every vertex in $S^* - (A \cup \{v\})$ is not adjacent to v , so $S^* - (A \cup \{v\})$ is a quasi-clique. Also A is $\cap_{s \in S \cup \{v\}} N(s)$, so $|E(G[A])| + |E(G; A, V(G) - (A \cup S \cup \{v\}))| \geq \frac{|A|^2 - 2|A|}{c}$. For every pair of non-adjacent vertices $\{x, y\}$, one of them is the supplement of quasi-edge associated to this pair, so quasi-edges associated to non-adjacent pairs in A and $S^* - (A \cup \{v\})$ are disjoint. With statements mentioned above we can conclude that:

$$|E(G[S^*])| + |E(G; S^*, V(G) - (S^* \cup S))| \geq \frac{|A|^2 - 2|A|}{c} + \binom{|S^*| - |A| - 1}{2} + |A|.$$

The right side of the inequality is a function of $|A|$, that we call it $f(|A|)$. One can find out that:

$$f'(|A|) = \frac{(c+2)|A|}{c} + \left(\frac{5}{2} - \frac{2}{c}\right) - |S^*|$$

So $f'(|A|)$ has negative value whenever $0 \leq |A| \leq \frac{2|S^*| - 4}{c}$ and $|S^*| \geq 3$. So it suffices to prove that $f(\frac{2|S^*| - 4}{c}) \geq \frac{|S^*|^2 - 2|S^*|}{c}$, which is done by Lemma A.2. On the other hand when $|A| \geq \frac{2|S^*| - 4}{c}$ by definition of A , we can easily conclude that:

$$|E(G[S^*])| \geq \frac{|A||S^*|}{2} \geq \frac{|S^*|^2 - 2|S^*|}{c}.$$

□

Lemma 3.3. Let $G = (V, E)$ be a 3_t -critical graph. If $v \in V$, then $V - N_G[v]$ is a quasi-clique.

Proof. This lemma is generalized from a lemma in ([13]), in which v was assumed as a vertex with minimum degree in G . Since the proof was independent of such assumption, the same proof is correct. \square

Now, we present the main result of this paper:

Theorem 3.4. *Suppose that $c = 2 + \sqrt{2}$, and a is the smallest root of the equation $(2c + 4)x^2 - 4cx + c = 0$, which is equal to $\frac{\sqrt{2}-\sqrt{2-\sqrt{2}}}{2} \approx 0.32442$. Let $G(V, E)$ be a 3_t -critical graph of order n , size m and minimum degree δ . If $n \geq 3$ and $\delta \leq an - 1$ then,*

$$m > \lceil \frac{n(n-2)}{4} \rceil$$

Proof. First, note that for every positive integer n :

- if n is even $n(n-2)$ is divisible by 4.
- if n is odd $n(n-2) + 1$ is divisible by 4.

So it suffices to prove that:

$$m > \frac{n(n-2) + 1}{4}$$

Let $v \in V(G)$ be a vertex with δ neighbors and $A = N_G(v)$. Also let $B = V - N_G[v]$, then by Lemma 3.3, B is a quasi-clique. Also by Lemma 3.2, $|E(G[A])| + |E(G; A, B)| \geq \frac{\delta^2 - 2\delta}{c}$. A and B are disjoint, so the quasi-edges associated to non-adjacent pairs in A are disjoint from the quasi-edges associated to non-adjacent pairs in B , because every quasi-edge has unique supplement. Therefore, we have:

$$m \geq \delta + \frac{\delta^2 - 2\delta}{c} + \binom{n-1-\delta}{2}$$

So by Lemma A.1 we have:

$$m > \frac{n(n-2) + 1}{4}$$

\square

Theorem 3.5. *For every diameter-2-critical graph G of order n and size m , if $\Delta(G) \geq 0.6756n$, then $m < \lfloor \frac{n^2}{4} \rfloor$*

Proof. Since $\text{diam}(G) = 2$, so $n \geq 3$. Let \bar{G} be complement of G . Assume that size of \bar{G} is m' . Since $m + m' = \binom{n}{2}$, so it suffices to prove that:

$$m' > \lceil \frac{n(n-2)}{4} \rceil.$$

We have:

$$\delta(\bar{G}) = n - 1 - \Delta(G) \leq 0.3244n - 1$$

Note that by Theorem 2.1, \bar{G} is either 3_t -critical or 4_t -supercritical. If \bar{G} is 4_t -supercritical, then by Theorem 2.2, \bar{G} is disjoint union of two non-trivial graph and size of the smaller one is less than $0.3244n - 1$, which means

$$m' \geq \binom{0.3244n - 1}{2} + \binom{0.6756n + 1}{2} > \lceil \frac{n(n-2)}{4} \rceil.$$

So we may consider that \bar{G} is 3_t -critical, which is shown in Theorem 3.4. \square

References

- [1] J. A. Bondy and U. S. R. Murty . Graph Theory.
- [2] Haynes TW, Hedetniemi ST, Slater PJ Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998
- [3] Haynes, Teresa W. Henning, Michael A. van der Merwe, Lucas C. Yeo, Anders Progress on the Murty-Simon Conjecture on diameter-2 critical graphs: a survey , Journal of Combinatorial Optimization 30 (2015), 579-595
- [4] D. Hanson and P. Wang, A note on extremal total domination edge critical graphs. Util. Math. 63 (2003), 89-96.
- [5] L. Caccetta and R. Haggkvist, On diameter critical graphs. Discrete Math. 28 (1979), no. 3, 223-229.
- [6] J. Plesnk, Critical graphs of given diameter. Acta F.R.N Univ Comen Math. 30 (1975), 7193.
- [7] G. Fan, On diameter 2-critical graphs. Discrete Math. 67 (1987), 235240.
- [8] J. Xu, A proof of a conjecture of Simon and Murty (in Chinese). J. Math. Res. Expo- sition 4 (1984), 8586.
- [9] Z. Füredi, The maximum number of edges in a minimal graph of diameter 2. J. Graph Theory 16 (1992), 8198.
- [10] T. W. Haynes, C. M. Mynhardt, and L. C. van der Merwe, Criticality index of total domination. Congr. Numer. 131 (1998), 6773.
- [11] T. W. Haynes, C. M. Mynhardt and L. C. van der Merwe, Total domination edge critical graphs. Utilitas Math. 54 (1998), 229240.
- [12] T. W. Haynes, M. A. Henning, L. C. van der Merwe, and A. Yeo, On a conjecture of Murty and Simon on diameter-2-critical graphs. Discrete Math. 311 (2011), 1918-1924.

- [13] T. W. Haynes, M. A. Henning, L. C. van der Merwe, and A. Yeo, A maximum degree theorem for diameter-2-critical graphs. Central European Journal of Mathematics 12 (2014), 1882-1889
- [14] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Editors, Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York (1998).
- [15] M. A. Henning, and A. Yeo, Total Domination in Graphs, Springer, (2013)

A

Proof of Inequalities

Lemma A.1. Suppose that $c = 2 + \sqrt{2}$, and a is smaller root of the equation $(2c + 4)x^2 - 4cx + c = 0$, which is equal to $\frac{\sqrt{2}-\sqrt{2-\sqrt{2}}}{2} \approx 0.3244$. If $an - 1 \geq y \geq 0$ and $n \geq 3$ then:

$$y + \frac{y^2 - 2y}{c} + \binom{n-1-y}{2} > \frac{n(n-2)+1}{4}$$

Proof. Let $f(y) = y + \frac{y^2 - 2y}{c} + \binom{n-1-y}{2}$. We have:

$$\begin{aligned} f'(y) &= 1 + \frac{2y-2}{c} - n + y + \frac{3}{2} \\ &= -n + \frac{5}{2} + \sqrt{2}y - \frac{2}{c} \\ &< -n + \frac{5}{2} + \sqrt{2}(an-1) - \frac{2}{c} < 0 \end{aligned}$$

Which means $f(y) \geq f(an-1)$. Let $g(n) = f(an-1) - \frac{n(n-2)+1}{4}$. Now it suffices to prove $g(n)$ has positive value for every $n \geq 3$.

$$g(n) = \frac{1}{4}((-8 + 7\sqrt{2} + 4\sqrt{4-2\sqrt{2}} - 7\sqrt{2-2\sqrt{2}})n + 6\sqrt{2} - 11)$$

So the coefficient of n is positive and $g(3) \approx 0.025 > 0$, so we can conclude that $g(n)$ is positive when $n \geq 3$. \square

Lemma A.2. Let $n \geq 3$ be a positive integer and $c = 2 + 2\sqrt{2}$, then

$$\frac{(\frac{2n-4}{c})^2 - 2(\frac{2n-4}{c})}{c} + \left(n - \left(\frac{2n-4}{c}\right) - 1\right) + \left(\frac{2n-4}{c}\right) \geq \frac{n^2 - 2n}{c}.$$

Proof. We prove that $f(n) = \frac{(\frac{2n-4}{c})^2 - 2(\frac{2n-4}{c})}{c} + \left(n - \left(\frac{2n-4}{c}\right) - 1\right) + \left(\frac{2n-4}{c}\right) - \frac{n^2 - 2n}{c}$

has positive value.

$$f(n) = \frac{1}{2}((3\sqrt{2} - 4)n + (8 - 6\sqrt{2}))$$

$$= \frac{(3\sqrt{2} - 4)}{2}(n - 2) > 0$$

So $f(n)$ is positive for $n \geq 3$.

□